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The propagation of impact-induced tensile waves in a kind of phase-transforming materials[☆]

Hui-Hui Dai^a, De-Xing Kong^{b,*}^a*Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*^b*Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200030, China*

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Dedicated to Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

This paper concerns the propagation of impact-generated tensile waves in a one-dimensional bar made of a kind of phase-transforming materials, for which the stress–strain curve changes from concave to convex as the strain increases. We use the fully nonlinear curve instead of approximating it by a tri-linear curve as often used in literature. The governing system of partial differential equations is quasi-linear and hyperbolic–elliptic. It is well known that the standard form of the initial-boundary value problem corresponding to impact is not well-posed at all levels of loading. In this paper, we describe in detail the propagation of impact-induced tensile waves for all levels. In particular, by means of the uniqueness condition on phase boundary derived recently, we construct a physical solution of the initial-boundary value problem mentioned above, and analyze the geometrical structure and behavior of the physical solution.

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1. Introduction

This paper concerns the propagation of impact-induced tensile waves in a one-dimensional bar made of a kind of phase-transforming materials. For a thin bar or rod made of this kind of materials and placed

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* Corresponding author. Tel.: +86 21 6248 0636.

E-mail address: dkong@sjtu.edu.cn (D.-X. Kong).

in an equilibrium state of uniaxial tension, the curve of the nominal stress and the longitudinal strain is concave for small to moderate strains but strongly convex for large strains, precisely speaking, the stress–strain curve has a peak-valley combination. A host of materials such as various polymers [15], natural rubbers [11] and metals (e.g., alloys) [14] can be regarded as the phase-transforming materials considered here. This kind of phase-transforming material has many applications, for example, they can be used to design rotary actuators, snake-like robots and delicate medical devices. Various aspects of these types of materials have been studied by many people (e.g., [5–8,12]). From the point of view of both mathematics and physics, the biggest problem is the uniqueness of solutions. Abeyaratne et al. [1] considered the impact-induced phase transition problem in a semi-infinite slab with a given velocity $-V$ at the end (cf. [1–4]). The mathematical model investigated by them are the one-dimensional dynamical equations

$$\rho v_t = \sigma_x, \quad \gamma_t = v_x, \quad (1)$$

where x and t are, respectively, the spatial and temporal variables, $\gamma = w_x$ is the axial strain and $v = w_t$ is the velocity (where w is the axial displacement), σ is the stress and ρ is the density. The system (1) is hyperbolic for a standard material for which $\sigma'(\gamma) > 0$, and is hyperbolic–elliptic for a typical phase-transforming material for which $\sigma'(\gamma)$ changes signs (usually the stress–strain curve has a peak-valley combination). For a phase-transforming material, the phase boundary is induced when the given velocity V is within a certain interval. Under the so-called tri-linear approximation for the stress–strain curve, Abeyaratne et al. [1] described the solution in the (t, x) -plane, which contains a shock wave and a phase boundary. Knowles [13] studied the impact-generated tensile waves in a one-dimensional semi-infinite bar made of a rubberlike material. For this kind of material, $\sigma'(\gamma) > 0$ but $\sigma(\gamma)$ has one and only one inflection point. He showed that (1) there are three regimes of response, depending on the intensity of the loading; (2) for weak impacts, corresponding to small impact speeds, the response is a pure centered rarefaction wave, as in a material with a concave stress–strain curve; (3) for the severest impacts, the response is a pure shock wave, as in materials with convex stress–strain relations; (4) for the intermediate range of impact velocity, the response typically exhibits a two-wave structure consisting of a centered rarefaction wave followed by a *phase boundary*. In the intermediate case, there is a one-parameter family of solutions to the initial-boundary problem. In order to select the unique *admissible solution*, Abeyaratne and Knowles introduced the concept of *driving force*, which is defined via the dissipation rate (cf. [2–4]), and the kinetic relations. They also used the hypotheses such as *maximally dissipative kinetics* or *dissipation-free kinetics*. In the literature, the kinetic relation is regarded as an extra constitutive relation for the material to be determined experimentally and is proposed to give an additional condition, besides the Rankine–Hugoniot conditions across the phase boundary, to obtain the unique solution. However, for these hypotheses, there is the lack of a physical basis and no mathematical justification (cf. [13]). For a slender circular cylinder, by taking into account the effect due to the radial deformation, Dai [9] established a proper model equation, and then, by using this model equation and matching its travelling wave solution to those in the outer regions, he obtained a uniqueness condition on phase boundary for the solutions.

This paper concerns the propagation of impact-generated tensile waves in a semi-infinite rod made of a kind of phase-transforming materials mentioned above with a given velocity $-V$ at the end. We shall construct the solutions for the loading at all levels. The novelty of the present work is that we use a fully nonlinear stress–strain curve, while in literature the solutions were only constructed for a “linearized”

tri-linear curve. Also, by means of the uniqueness condition on phase boundary derived by Dai [9], we construct the unique physical solution of the initial-boundary value problem, and analyze the geometrical structure and behavior of the physical solution.

The paper is organized as follows. In Section 2, we state the initial-boundary value problem modeling the impact-generated tensile waves in a semi-infinite rod made of the above-mentioned phase-transforming material, and briefly recall the uniqueness condition derived by Dai [9]. In Section 3, we elucidate the structure of solutions in each of the three regimes of impact velocity alluded to above. Moreover, the geometrical structure and behavior of the solutions are also described in this section. The key Lemma 3.1 is proved in the last section, Section 4.

2. The initial-boundary value problem

In its stress-free reference state, the one-dimensional bar considered here occupies the nonnegative x -axis. After impact, a particle located at x in the reference state is carried out to $x + u(t, x)$, where u stands for the longitudinal displacement of the particle at time t . The strain $\gamma(t, x)$ and particle velocity $v(t, x)$ are defined by $\gamma = u_x$ and $v = u_t$, respectively. To ensure that the mapping $x \rightarrow x + u(t, x)$ is one-to-one, we assume that $\gamma > -1$. The nominal stress at time t at this particle is denoted by $\sigma(t, x)$. In Lagrangian description, of the motion, balance of linear momentum and kinematic compatibility are equivalent to the system of partial differential equations of conservation laws

$$\rho v_t = \sigma_x, \quad \gamma_t = v_x, \quad (1)$$

where γ, v are smooth, and jump conditions

$$\dot{s}[\gamma] + [v] = 0, \quad \dot{s}\rho[v] + [\sigma] = 0 \quad (2)$$

at a moving strain discontinuity whose referential location is $x = s(t)$ at time t . Here $[f] = f(t, s(t)+) - f(t, s(t)-)$, ρ is the constant mass per unit referential volume, and $\dot{s} = s'(t)$ is the Lagrangian velocity of the discontinuity.

The material of the bar is taken to be elastic so that

$$\sigma = \sigma(\gamma) \quad \text{for } \gamma > -1, \quad (3)$$

where the stress-response function $\sigma(\gamma)$ is given and assumed twice continuously differentiable. We point out that the theory is purely mechanical rather than thermomechanical, the thermal effects are omitted. For the material characterized by Eq. (3), the field Eq. (1) becomes

$$\rho v_t = \sigma'(\gamma)\gamma_x, \quad \gamma_t = v_x. \quad (4)$$

The system (4) is hyperbolic for a standard material for which $\sigma'(\gamma) > 0$, and is hyperbolic–elliptic for a phase-transforming material for which $\sigma'(\gamma)$ changes signs (usually the stress–strain curve has a peak–valley combination).

For a phase-transforming material, the discontinuity may be a phase boundary. If the discontinuity is a phase boundary, then the situation is quite different. In this case, the influence of the radial deformation in the axial direction is dominant and cannot be neglected. Taking into account the effect due to the radial

deformation, Dai [9] established a proper model equation, and obtained a uniqueness condition for the solution

$$\int_{\gamma^-}^{\gamma^+} \sigma(\gamma) d\gamma = \gamma^+ \sigma(\gamma^+) - \gamma^- \sigma(\gamma^-) - \frac{\rho}{2} \dot{s}^2 (\gamma^{+2} - \gamma^{-2}). \quad (5)$$

In this paper, we consider a kind of phase-transforming material, whose stress-response function $\sigma(\gamma)$ satisfies the following properties:

- (i) $\sigma(\gamma) \in C^2([-1, \infty))$ and $\sigma(0) = 0$;
- (ii) let α^1, β^1 be assigned constants consistent with $0 < \alpha^1 < \beta^1 < \infty$, it holds that

$$\sigma'(\gamma) > 0 \text{ on } [0, \alpha^1) \cup (\beta^1, \infty), \quad \sigma'(\gamma) < 0 \text{ on } (\alpha^1, \beta^1); \quad (6)$$

- (iii) there is a constant $\gamma^* \in (\alpha^1, \beta^1)$ such that

$$\sigma''(\gamma) < 0 \text{ on } [0, \gamma^*), \quad \sigma''(\gamma) > 0 \text{ on } (\gamma^*, \infty); \quad (7)$$

The stress-response function restricted to the interval $[0, \alpha^1]$ is called the α -branch; the restriction of the stress-response function to the interval $[\beta^1, \infty)$ is called the β -branch.

To mimic the behavior, we consider a bar made of stress-response function given explicitly by

$$\sigma(\gamma) = E \left(\frac{\gamma^3}{3} - \frac{1}{2}(\alpha + \beta)\gamma^2 + \alpha\beta\gamma \right), \quad (8)$$

where E is the Young's modulus for infinitesimal strains, α and β are constants with

$$0 < \alpha < \beta < \infty. \quad (9)$$

Obviously, $\sigma(\gamma)$ is concave for $-1 < \gamma < (\alpha + \beta)/2$ and convex for $\gamma > (\alpha + \beta)/2$; see Fig. 1.

The sound-wave speed $c(\gamma)$ for the bar governed by Eq. (3) is given by

$$c(\gamma) = \sqrt{\sigma'(\gamma)/\rho} = c_0 \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} \quad \text{for } \gamma \in [0, \alpha] \cup [\beta, \infty), \quad (10)$$

where c_0 is the speed of small amplitude waves at the undeformed state

$$c_0 = c(0) = \sqrt{E/\rho}. \quad (11)$$

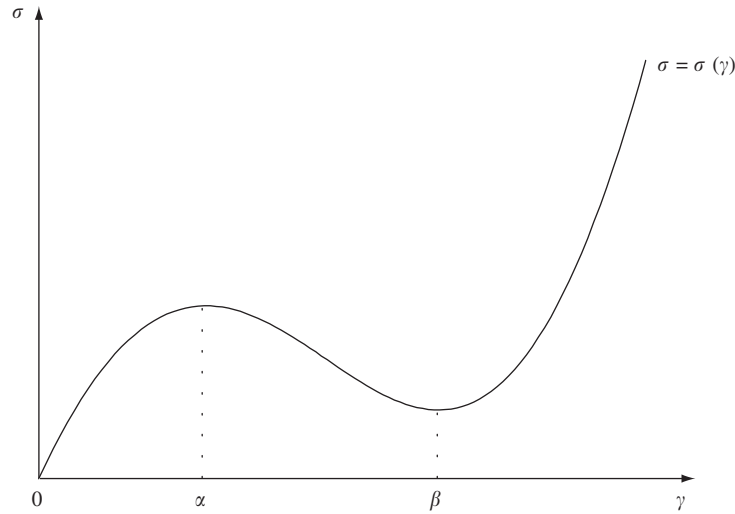
Impact problem: The bar is assumed to be initially at rest in its undeformed state; at time $t = 0$, a particle velocity $v = -V$ (V is a positive constant) is imposed at the end $x = 0$ and maintained for all subsequent time. Mathematically speaking, this corresponds to the following initial-boundary conditions

$$\gamma(0, x) = 0, \quad v(0, x) = 0 \quad \text{for } x > 0, \quad (12)$$

$$v(t, 0) = -V \quad \text{for } t > 0. \quad (13)$$

We seek the solution of the impact problem (4), (12)–(13) on the first quadrant.

As in [13], here we consider only tensile impact so that we assume throughout that $V > 0$. Because the impact problem does not involve a parameter with the dimension of either length or time, it is invariant under a change of scale $x \rightarrow kx, t \rightarrow kt$. Therefore, we only consider solutions that also have this

Fig. 1. Graph of the stress-response function $\sigma(\gamma)$.

invariance, these are necessarily functions of $\xi = x/t$ only. For such a scale-invariant solution, the first quadrant of the (t, x) -plane is divided into wedges issuing from the origin, on each of which the solution is either constant or takes the form of a fan.

The dissipation associated with the shock waves or phase boundaries to be encountered below plays a significant role. Let γ, v be a solution of the system (4) that is smooth on a subinterval $[x_1, x_2]$ of the positive x -axis except at a moving discontinuity in γ and v which is located at $x = s(t)$ at t . Let $D(t)$ be the rate of dissipation at time t , then

$$D(t) = \sigma(t, x)v(t, x)|_{x_1}^{x_2} - \frac{d}{dt} \int_{x_1}^{x_2} \{\rho v^2(t, x)/2 + W(\gamma(t, x))\} dx, \quad (14)$$

where

$$W(\gamma) = \int_0^\gamma \sigma(\gamma) d\gamma \quad (15)$$

is the stored elastic energy per unit reference volume at the strain γ . If the rate of dissipation $D(t)$ has an alternate representation

$$D(t) = f(t)s'(t) \geq 0 \quad \text{for } t > 0, \quad (16)$$

then $f(t)$ is called the *driving force* per unit cross-sectional area acting at time t on the moving strain discontinuity (see [13]). This notion is directly related to the concept of *thermodynamic driving force* that plays a major role in materials science in connection with phase transformations.

For a phase-transforming material characterized by Eq. (8), direct calculation shows that

$$f = \frac{E}{12}(\gamma^+ - \gamma^-)^3\{(\alpha + \beta) - (\gamma^+ + \gamma^-)\}. \quad (17)$$

On the other hand, if the discontinuity is a phase boundary, then the uniqueness condition (5) becomes

$$\gamma^+ + \gamma^- = \alpha + \beta \quad (18)$$

for the phase-transforming material characterized by Eq. (8).

3. The impact-generated tensile waves

In this section, we elucidate the structure of solutions of the impact problem (4), (12)–(13) on the first quadrant.

Weak impacts: As in [13], we first examine the case that the impact speed V is small enough to keep the resulting maximum strain below the level $\gamma = \alpha$. Setting $\xi = x/t$, we look for a solution in the form

$$(v(t, x), \gamma(t, x)) = \begin{cases} (-V, \gamma_1) & \text{for } 0 \leq \xi \leq \xi_1, \\ (\hat{v}(\xi), \hat{\gamma}(\xi)) & \text{for } \xi_1 \leq \xi \leq \xi_2, \\ (0, 0) & \text{for } \xi \geq \xi_2, \end{cases} \quad (19)$$

where $\gamma_1 = \hat{\gamma}(\xi_1)$ and ξ_1, ξ_2 are to be determined; see Fig. 2.

The following argument is standard. Substituting formula (19) into system (4) gives

$$c(\hat{\gamma}(\xi)) = \xi, \quad \hat{v}'(\xi) = -\xi \hat{\gamma}'(\xi), \quad (20)$$

where $c(\gamma)$ is the sound-wave speed defined by Eq. (10). Continuity and the initial condition yields $\xi_2 = c_0$. Bearing in mind, the values of $\hat{\gamma}(\xi)$ must lie in $[0, \alpha]$. The vanishing of $\hat{v}(\xi)$ at $\xi = \xi_2 = c_0$ and the second equation in formulas (20) furnishes

$$\hat{v}(\xi) = \int_{\xi}^{c_0} \xi \hat{\gamma}'(\xi) d\xi. \quad (21)$$

The further continuity requirement $\hat{v}(\xi_1) = -V$ leads to

$$\int_{\xi_1}^{c_0} \xi \hat{\gamma}'(\xi) d\xi = -V, \quad (22)$$

i.e.,

$$\int_0^{\gamma_1} c(\gamma) d\gamma = V. \quad (23)$$

Thus, γ_1 is determined by Eq. (23). Once γ_1 is known, ξ_1 is found from the first equation in formulas (20).

Since $\xi_1 < c_0$, one has $\gamma_1 = \hat{\gamma}(\xi_1) > 0$ so that Eq. (23) has no solution if $V < 0$. When $V > 0$, the maximum strain in the rarefaction wave is the value γ_1 defined in Eq. (27). In order to keep $\gamma_1 \leq \alpha$, the impact velocity V must satisfy $V \leq V_*$, where

$$V_* = \int_0^{\alpha} c(\gamma) d\gamma. \quad (24)$$

The interval $(0, V_*]$ is called the regime of weak impacts.

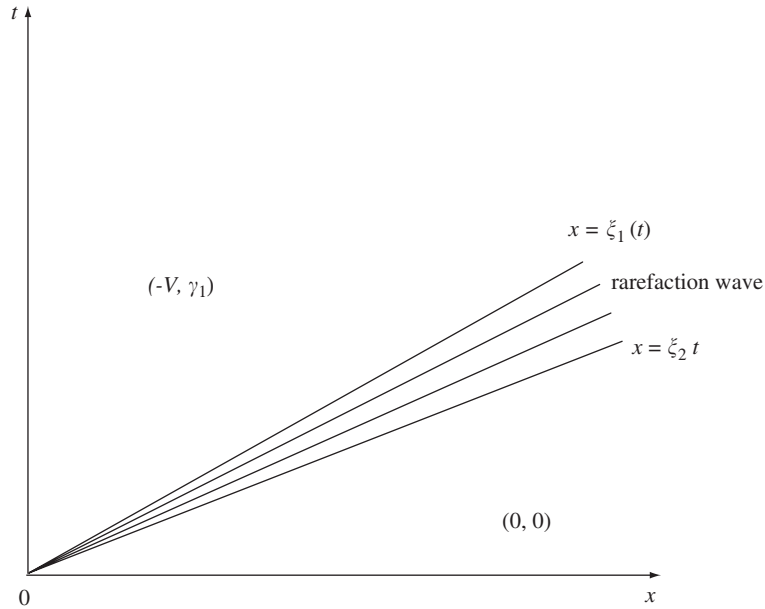


Fig. 2. Solution in the form of a centered rarefaction wave.

For the material characterized by Eq. (8), using Eq. (10) in the first equation in formulas (20), we find the strain in the rarefaction wave explicitly as

$$\gamma = \hat{\gamma}(x/t) = \frac{1}{2} \left[(\alpha + \beta) - \sqrt{(\alpha + \beta)^2 - 4(1 - \xi^2/c_0^2)\alpha\beta} \right]. \quad (25)$$

On the other hand, it follows from Eq. (10) and formula (24) that the critical impact velocity V_* is

$$V_* = \frac{c_0}{2\sqrt{\alpha\beta}} \left\{ \frac{\alpha + \beta}{2} \sqrt{\alpha\beta} + \left(\frac{\beta - \alpha}{2} \right)^2 \ln \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}} \right\}. \quad (26)$$

One can show that $\hat{\gamma}''(\xi) < 0$, so that the strain is a concave function of ξ in the rarefaction wave.

Remark 3.1. In the present situation, i.e., $0 < V \leq V_*$, we can prove that the solution of the impact problem (4), (12)–(13) on the first quadrant only contains a rarefaction wave, but no shock wave or phase boundary.

Strong impacts: We next consider the case of a large impact speed V , postponing the more complex case in which V is in an intermediate range. Since one might expect behavior similar to that of a material with a convex stress–strain curve, it is nature to seek a solution with a shock, with the bar jumping directly from the unstressed state to a severely strained configuration behind the shock. In this case, we seek a solution with the form of a shock wave

$$(v(t, x), \gamma(t, x)) = \begin{cases} (-V, \gamma^-) & \text{for } 0 \leq x < \dot{s}t, \\ (0, 0) & \text{for } x > \dot{s}t, \end{cases} \quad (27)$$

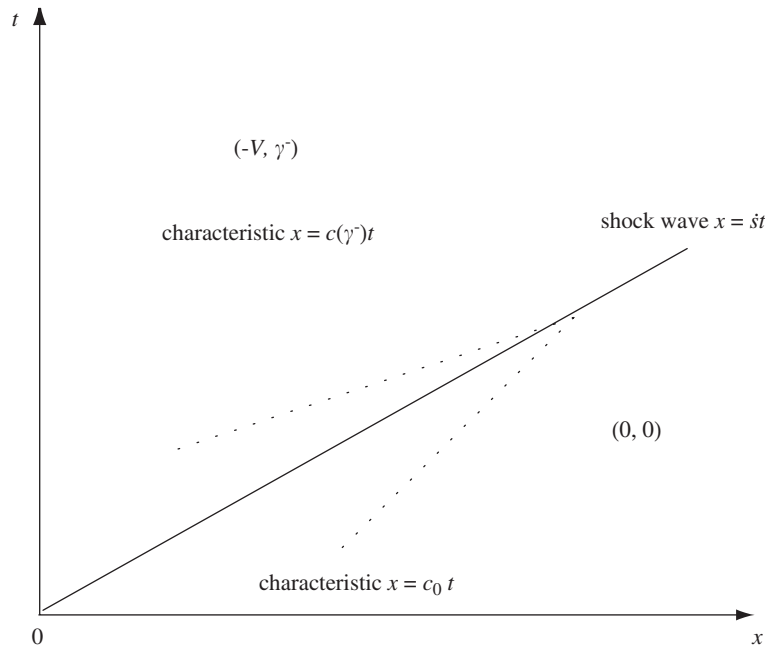


Fig. 3. Solution in the form of a shock wave.

where the back-state strain γ^- and the constant Lagrangian shock velocity \dot{s} are to be determined; see Fig. 3.

Since V and γ^- are constants, the system (4) is obviously satisfied, only the jump conditions remain. These specialize to

$$-\gamma^- \dot{s} + V = 0, \quad -\sigma(\gamma^-) + \rho \dot{s} V = 0. \quad (28)$$

Eliminating \dot{s} from these yields the equation to determine γ^-

$$\gamma^- \sigma(\gamma^-) = \rho V^2. \quad (29)$$

For the special material (8), this becomes

$$\gamma^- c_0 \sqrt{\gamma^{-2}/3 - (\alpha + \beta)\gamma^-/2 + \alpha\beta} = V \sqrt{\alpha\beta}. \quad (30)$$

Once γ^- is determined in terms of V , the shock speed is found from

$$\dot{s} = \frac{V}{\gamma^-} = \frac{c_0 \sqrt{\gamma^{-2}/3 - (\alpha + \beta)\gamma^-/2 + \alpha\beta}}{\sqrt{\alpha\beta}}. \quad (31)$$

Since the strain γ^+ ahead of the shock wave vanishes, the driving force (17) becomes

$$f = E\gamma^{-3}[\gamma^- - (\alpha + \beta)]/12. \quad (32)$$

Because $\dot{s} > 0$, the dissipation inequality (16) require that $f \geq 0$, and hence

$$\gamma^- \geq \alpha + \beta \quad (\text{physical admissibility}). \quad (33)$$

The left-hand side of Eq. (30) makes sense and is a monotonically increasing function of γ^- for $\gamma^- > 3(\alpha + \beta)/2$ that takes all values at least as great as $3(\alpha + \beta)c_0/2$. It follows that Eq. (30) has a unique solution γ^- corresponding to a physically admissible shock wave, provided that

$$V > 3(\alpha + \beta)c_0/2 \triangleq V_{***}. \quad (34)$$

Since

$$\gamma^- > 3(\alpha + \beta)/2, \quad (35)$$

it follows from Eq. (31) that

$$\dot{s} > c_0. \quad (36)$$

On the other hand, noting Eq. (31) and

$$c(\gamma^-) = c_0 \sqrt{(\gamma^- - \alpha)(\gamma^- - \beta)/\alpha\beta}, \quad (37)$$

we have the following fact:

$$c(\gamma^-) > \dot{s}, \quad (38)$$

if and only if

$$\gamma^- > 3(\alpha + \beta)/4. \quad (39)$$

Therefore, in the case that condition (35) is satisfied, $x = \dot{s}t$ is indeed a shock wave in the classical sense.

Obviously, in those pure-shock-wave solutions just constructed for which $\gamma^- > 3(\alpha + \beta)/2$ and therefore $V > V_{***}$, the shock wave is *supersonic* in the Lagrangian sense with respect to the undisturbed state ahead of it. As in [13], this property leads us to define the regime of *strong impact* as that corresponding to impact velocities for which $V > V_{***}$.

Particularly, when the back-state strain takes the value $\gamma^- = 3(\alpha + \beta)/2$, it follows from Eqs. (30)–(31) that the Lagrangian shock wave speed and the impact velocity take the respective values $\dot{s} = c_0$ and $V = V_{***}$. In this case, the solution is still in the form (27), where γ^- , \dot{s} are also defined by Eqs. (30) and (31), respectively; see Fig. 4. However, the discontinuity $x = st$ is a *degenerate* shock wave, because by Eqs. (30) and (31) we have

$$\dot{s} = c_0 \quad \text{and} \quad \dot{s} < c(\gamma^-). \quad (40)$$

Of course, in the present situation, the physical admissible condition (33) still holds.

Impact of intermediate strength ($V_* < V < V_{***}$): With the hope of filling the gap $V_* < V < V_{***}$, where as yet no solution is obtained, we next attempt to construct the solution of the impact problem (4), (12)–(13).

In what follows, for simplicity, we assume

$$\beta < 3\alpha, \quad (41)$$

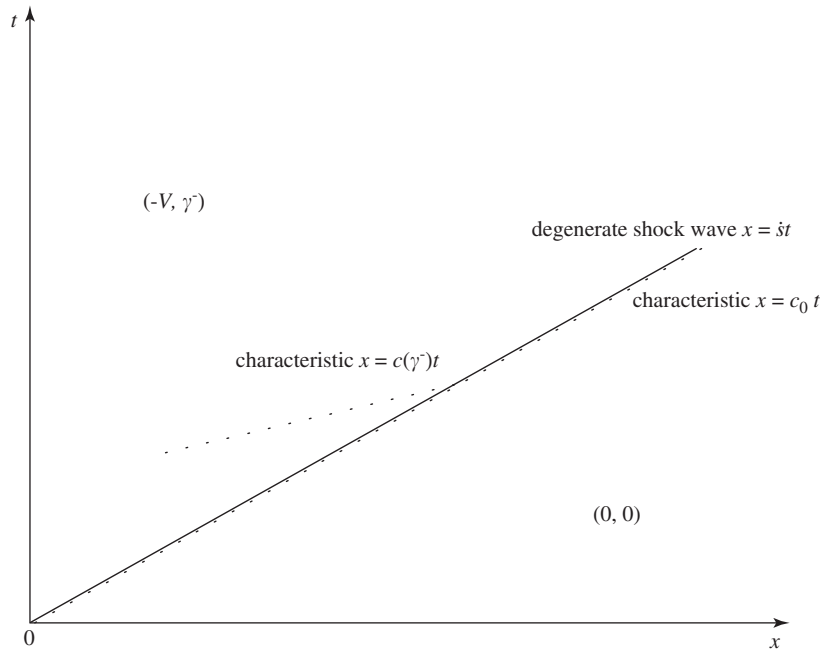


Fig. 4. Solution in the form of a degenerate shock wave.

which together with assumptions (9) yields that

$$\sigma(\gamma) > 0 \quad \text{for } \gamma > 0. \quad (42)$$

This means that the stress-response function $\sigma(\gamma)$ is positive for strain $\gamma > 0$.

Let

$$V_{**} = \frac{c_0}{\sqrt{\alpha\beta}}(\alpha + \beta)\sqrt{\alpha\beta - (\alpha + \beta)^2/6}. \quad (43)$$

The following lemma will be proved in Section 4.

Lemma 3.1. *Under the assumptions (9) and (41), V_* , V_{**} , V_{***} are positive constants and satisfy*

$$0 < V_* < V_{**} < V_{***} < \infty. \quad (44)$$

There are two cases:

Case 1 ($V_{**} \leq V < V_{***}$): In the present situation, by assumption (41), the left-hand side of Eq. (30) makes sense and is a strictly monotonically increasing function of γ^- for $\gamma^- > \alpha + \beta$ that takes all values at least as great as $(\alpha + \beta)c_0\sqrt{\alpha\beta - (\alpha + \beta)^2/6}$. It follows that Eq. (30) has a unique solution γ^- corresponding to a physically admissible discontinuity, provided that $V \in [V_{**}, V_{***})$. As the impact velocity V increases, the back-state strain γ^- as determined by Eq. (30) also increases, and then the discontinuity speed \dot{s} is

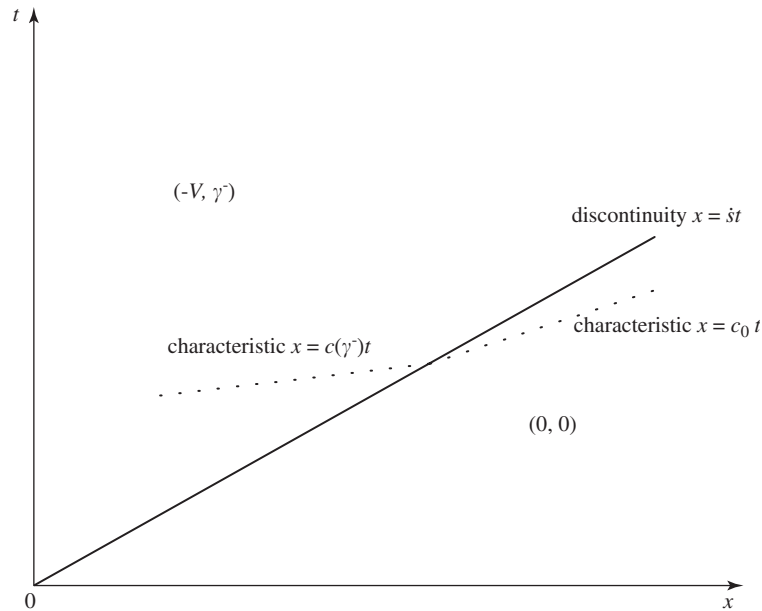


Fig. 5. Physical admissible solution.

also an increasing function of V . Thus, the *physical admissible solution*¹ is in the following form:

$$(v(t, x), \gamma(t, x)) = \begin{cases} (-V, \gamma^-) & \text{for } 0 \leq x < \dot{s}t, \\ (0, 0) & \text{for } x > \dot{s}t, \end{cases} \quad (45)$$

where γ^- , \dot{s} are still defined by Eqs. (30) and (31), respectively. However, the discontinuity $x = \dot{s}t$ is not a shock wave in the classical sense (see Fig. 5). In fact, in this case, the condition $V_{**} \leq V < V_{***}$ corresponds to

$$\alpha + \beta \leq \gamma^- < 3(\alpha + \beta)/2. \quad (46)$$

Hence, it follows from Eq. (31) that, when $\gamma^- \in [\alpha + \beta, 3(\alpha + \beta)/2)$,

$$\dot{s} < c_0 \quad \text{and} \quad \dot{s} < c(\gamma^-) = c_0 \sqrt{(\gamma^- - \alpha)(\gamma^- - \beta)/\alpha\beta}, \quad (47)$$

provided that $V_{**} \leq V < V_{***}$ (equivalently, $\alpha + \beta \leq \gamma^- < 3(\alpha + \beta)/2$).

Case 2 ($V_* < V < V_{**}$): We finally examine the case that the impact speed V is in the open interval (V_*, V_{**}) . We now attempt to construct one solution that involves both a centered rarefaction wave and a *phase boundary*, as shown in Fig. 6. Thus, we propose that γ and v have the following two-waves form

$$(v(t, x), \gamma(t, x)) = \begin{cases} (-V, \gamma^-) & \text{for } 0 \leq x < \dot{s}t, \\ (v^+, \gamma^+) & \text{for } \dot{s}t < x \leq \xi_1 t, \\ (\hat{v}(x/t), \hat{\gamma}(x/t)) & \text{for } \xi_1 t \leq x \leq \xi_2 t, \\ (0, 0) & \text{for } x \geq \xi_2 t, \end{cases} \quad (48)$$

where γ^- , \dot{s} , ξ_1 , ξ_2 , \hat{v} and $\hat{\gamma}$ are to be determined and, by continuity, $\gamma^+ = \hat{\gamma}(\xi_1)$ and $v^+ = \hat{v}(\xi_1)$.

¹ See [13] for the definition.

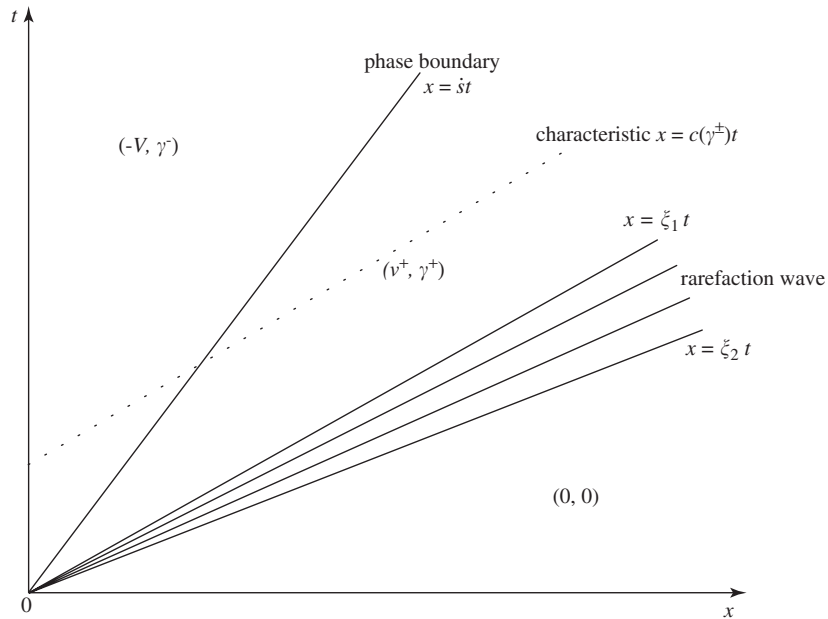


Fig. 6. Solution with a centered rarefaction wave and a phase boundary.

As in the centered rarefaction wave arising from weak impact, we have $\xi_2 = c_0$; since the phase boundary is to trail the centered rarefaction wave, we have to require $\dot{s} \leq \xi_1$. Assuming that γ and v are continuous everywhere except across the phase boundary, then we have $\xi_1 = c(\gamma^+)$ so that

$$\dot{s} \leq c(\gamma^+). \quad (49)$$

Eq. (20) for the centered rarefaction wave holds in the present situation as well. Since $\hat{v}(\xi_2) = 0$, we observe from Eq. (20) that

$$v^+ = \hat{v}(\xi_1) = \int_{\xi_1}^{c_0} \xi \hat{\gamma}'(\xi) d\xi = - \int_0^{\gamma^+} c(\gamma) d\gamma. \quad (50)$$

Using Eq. (51) in the jump condition (2) applied at the phase boundary $x = \dot{s}t$ leads to

$$(\gamma^+ - \gamma^-)\dot{s} - \left\{ \int_0^{\gamma^+} c(\gamma) d\gamma - V \right\} = 0, \quad (51)$$

$$\sigma(\gamma^+) - \sigma(\gamma^-) - \rho\dot{s} \left\{ \int_0^{\gamma^+} c(\gamma) d\gamma - V \right\} = 0. \quad (52)$$

These are two equations for the front- and back-state strains γ^\pm at the phase boundary; they involve the phase boundary speed \dot{s} as an unknown function of V , while the impact velocity V as a given datum. In this case, we make use of the uniqueness condition 5 derived by [9], we then have the third relation between γ^\pm and \dot{s} .

Eliminating the common contents of the braces between Eqs. (51) and (52) gives the standard formula relating the phase boundary speed and the slope of the chord connecting the two points on the stress–strain curve that correspond to the front- and back-states of the phase boundary

$$\rho \dot{s}^2 = \frac{\sigma(\gamma^+) - \sigma(\gamma^-)}{\gamma^+ - \gamma^-}. \quad (53)$$

Obviously, the Eqs. (51)–(52) are equivalent to either Eq. (51) or (52) together with Eq. (53).

For the special material (8), Eqs. (51) and (53) take the respective forms

$$(\gamma^+ - \gamma^-)\dot{s} + c_0 \int_{\gamma^+}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* = 0, \quad (54)$$

$$\rho \dot{s}^2 = E\{(\gamma^{+2} + \gamma^- \gamma^+ + \gamma^{-2})/3 - (\alpha + \beta)(\gamma^+ + \gamma^-)/2 + \alpha\beta\}; \quad (55)$$

while the uniqueness condition (5) becomes

$$\gamma^+ + \gamma^- = \alpha + \beta. \quad (56)$$

In what follows, we will show that, under the assumptions (9) and (41), the Eqs. (54)–(56) have a solution $(\gamma^+, \gamma^-, \dot{s})$ satisfying

$$\gamma^+ \in (0, \alpha) \quad \text{and} \quad \dot{s} > 0. \quad (57)$$

Let the right-hand side of Eq. (54) be F , namely,

$$\begin{aligned} F &= E\{(\gamma^{+2} + \gamma^- \gamma^+ + \gamma^{-2})/3 - (\alpha + \beta)(\gamma^+ + \gamma^-)/2 + \alpha\beta\} \\ &= E\{\gamma^{+2}/3 - (\alpha + \beta)\gamma^+/3 - (\alpha + \beta)^2/6 + \alpha\beta\} \triangleq \mathcal{F}(\gamma^+). \end{aligned} \quad (58)$$

By the definition of $\mathcal{F}(\gamma^+)$, we have

$$\mathcal{F}'(\gamma^+) < 0, \quad \forall \gamma^+ \in (0, \alpha), \quad (59)$$

$$\mathcal{F}(0) = E[\alpha\beta - (\alpha + \beta)^2/6] > 0 \quad (60)$$

and

$$\mathcal{F}(\alpha) = -E(\alpha - \beta)^2/6 < 0. \quad (61)$$

In the inequality (60), we have made use of the assumption (41). It follows from inequalities (59)–(61) that there exists a $\alpha_* \in (0, \alpha)$ such that

$$\mathcal{F}(\alpha_*) = 0 \quad \text{and} \quad \mathcal{F}(\gamma^+) > 0, \quad \forall \gamma^+ \in (0, \alpha_*). \quad (62)$$

Using the uniqueness condition (56) in Eqs. (54) and (55) leads to

$$(\alpha + \beta - 2\gamma^+)\dot{s} = c_0 \int_{\gamma^+}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_*, \quad (63)$$

$$\rho \dot{s}^2 = \mathcal{F}(\gamma^+). \quad (64)$$

We now restrict us to consider the case $\gamma^+ \in (0, \alpha_*)$. In this case, the right-hand side of Eq. (64) is positive (see (62)). Thus, by Eqs. (63) and (64) we have

$$\dot{s} = \left\{ c_0 \int_{\gamma^+}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* \right\} / (\alpha + \beta - 2\gamma^+), \quad (65)$$

$$\dot{s} = \sqrt{\mathcal{F}(\gamma^+)/\rho}. \quad (66)$$

By Eqs. (65) and (66), we find that

$$\left\{ c_0 \int_{\gamma^+}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* \right\} / (\alpha + \beta - 2\gamma^+) - \sqrt{\mathcal{F}(\gamma^+)/\rho} = 0. \quad (67)$$

Next, we show that the Eq. (67) has a solution $\gamma^+ \in (0, \alpha_*) \subseteq (0, \alpha)$. To do so, we denote the left-hand side of Eq. (67) by $G(\gamma^+)$, i.e.,

$$G(\gamma^+) = \left\{ c_0 \int_{\gamma^+}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* \right\} / (\alpha + \beta - 2\gamma^+) - \sqrt{\mathcal{F}(\gamma^+)/\rho}. \quad (68)$$

On the one hand, since $V \in (V_*, V_{**})$,

$$\begin{aligned} G(0) &= \left\{ c_0 \int_0^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* \right\} / (\alpha + \beta) - \sqrt{E\rho^{-1}[\alpha\beta - (\alpha + \beta)^2/6]} \\ &= (V - V_{**})/(\alpha + \beta) < 0. \end{aligned} \quad (69)$$

On the other hand, for every $V \in (V_*, V_{**})$,

$$G(\alpha_*) = \left\{ c_0 \int_{\alpha_*}^{\alpha} \sqrt{(\gamma - \alpha)(\gamma - \beta)/\alpha\beta} d\gamma + V - V_* \right\} / (\alpha + \beta - 2\alpha_*) > 0. \quad (70)$$

Combining inequalities (69) and (70), we find that there exists a $\gamma^+ \in (0, \alpha_*)$ such that Eq. (67) holds.

Once γ^+ has been solved, γ^- comes from Eq. (56) immediately, while \dot{s} can be determined directly by Eq. (65) or (66).

Finally, we examine the behavior of characteristics in the front- and back-states of the phase boundary. Using formula (10) and noting Eq. (56), we find

$$c(\gamma^+) = c(\gamma^-) = c_0 \sqrt{(\gamma^{\pm} - \alpha)(\gamma^{\pm} - \beta)/\alpha\beta} = \sqrt{E(\gamma^{\pm} - \alpha)(\gamma^{\pm} - \beta)/\rho}. \quad (71)$$

On the other hand, by Eq. (66), we have

$$\dot{s} = \sqrt{E[\gamma^{+2}/3 - (\alpha + \beta)\gamma^+/3 - (\alpha + \beta)^2/6 + \alpha\beta]\rho}. \quad (72)$$

A direct calculation gives

$$c^2(\gamma^{\pm}) - \dot{s}^2 = E[2\gamma^+ - (\alpha + \beta)]^2/6\rho > 0. \quad (73)$$

This implies that

$$c(\gamma^\pm) > \dot{s}. \quad (74)$$

See Fig. 6.

Remark 3.2. We would like to emphasize that, because of the formula (71), the characteristic speeds in the front- and back-states of the phase boundary are equal. This implies that the characteristic speeds keep continuous across the phase boundary. This property can be regarded as a geometrical character of the phase boundary under the uniqueness condition (56); see Fig. 6.

Remark 3.3. Using the method employed in [10], we may discuss the global structure stability of the physical solution constructed in this section. Because of space limitations, we omit the details.

4. Proof of Lemma 3.1

The positivity of V_* comes from the definition (24) directly.

We next show

$$V_* < V_{**}. \quad (75)$$

By formulas (26) and (43), it suffices to prove

$$\frac{c_0}{\sqrt{\alpha\beta}}(\alpha + \beta)\sqrt{\alpha\beta - \frac{(\alpha + \beta)^2}{6}} > \frac{c_0}{2\sqrt{\alpha\beta}} \left\{ \frac{\alpha + \beta}{2}\sqrt{\alpha\beta} + \left(\frac{\beta - \alpha}{2}\right)^2 \ln \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}} \right\}, \quad (76)$$

i.e.,

$$2(\alpha + \beta)\sqrt{\alpha\beta - \frac{(\alpha + \beta)^2}{6}} - \frac{\alpha + \beta}{2}\sqrt{\alpha\beta} - \left(\frac{\beta - \alpha}{2}\right)^2 \ln \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}} > 0. \quad (77)$$

Noting that

$$\left(\frac{\beta - \alpha}{2}\right)^2 \ln \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}} < 0, \quad (78)$$

in order to prove inequality (77), we only need to show that

$$2(\alpha + \beta)\sqrt{\alpha\beta - \frac{(\alpha + \beta)^2}{6}} - \frac{\alpha + \beta}{2}\sqrt{\alpha\beta} > 0. \quad (79)$$

Clearly, inequality (79) holds if and only if

$$4\sqrt{\alpha\beta - \frac{(\alpha + \beta)^2}{6}} > \sqrt{\alpha\beta}, \quad (80)$$

namely,

$$16\alpha\beta - 16(\alpha + \beta)^2/6 > \alpha\beta, \quad (81)$$

i.e.,

$$15\alpha\beta - 8(\alpha + \beta)^2/3 > 0. \quad (82)$$

In fact, the inequality (82) is valid for $\beta \in (\alpha, 3\alpha)$. Thus, the desired inequality (75) is true.

We finally prove

$$V_{**} < V_{***}. \quad (83)$$

Define

$$g(\gamma) = \frac{c_0\gamma}{\sqrt{\alpha\beta}} \sqrt{\frac{\gamma^2}{3} - \frac{(\alpha + \beta)\gamma}{2} + \alpha\beta} \quad \text{for } \gamma \geq (\alpha + \beta). \quad (84)$$

Noting the assumption (41), we find that the right-hand side of Eq. (84) makes sense and is positive for $\gamma \geq (\alpha + \beta)$. By a direct calculation, we have

$$g'(\gamma) = \frac{c_0}{2\sqrt{\alpha\beta}} \frac{4\gamma^2/3 - 3(\alpha + \beta)\gamma/2 + 2\alpha\beta}{\sqrt{\gamma^2/3 - (\alpha + \beta)\gamma/2 + \alpha\beta}}. \quad (85)$$

Let

$$h(\gamma) = 4\gamma^2/3 - 3(\alpha + \beta)\gamma/2 + 2\alpha\beta. \quad (86)$$

A direct calculation gives

$$h'(\gamma) = 8\gamma/3 - 3(\alpha + \beta)/2 > 0, \quad \forall \gamma \geq \alpha + \beta. \quad (87)$$

On the other hand,

$$h(\alpha + \beta) = -(\alpha + \beta)^2/6 + 2\alpha\beta. \quad (88)$$

Clearly, when $\beta \in (\alpha, 3\alpha)$, we have

$$h(\alpha + \beta) > 0. \quad (89)$$

Combining inequalities (87) and (89) yields that

$$h(\gamma) \geq h(\alpha + \beta) > 0, \quad \forall \gamma \geq \alpha + \beta. \quad (90)$$

Using inequality (90) in formula (85) leads to

$$g'(\gamma) > 0, \quad \forall \gamma \geq \alpha + \beta. \quad (91)$$

It follows that

$$g(3(\alpha + \beta)/2) > g(\alpha + \beta). \quad (92)$$

This is the desired inequality (83). Thus, the proof of Lemma 3.1 is completed.

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